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FURTHER COMPARISON OF STOCHASTIC AND DETERMINISTIC MODELS FOR THE OPTIMAL CONTROL OF LANCHESTER-TYPE ATTRITION PROCESSES

William Pickens Hannah

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THESIS

FURTHER COMPARISON OF STOCHASTIC AND DETERMINISTIC MODELS FOR THE OPTIMAL CONTROL OF LANCHESTER-TYPE ATTRITION PROCESSES

by

William Pickens Hannah

September 1974

Thesis Advisor: James G. Taylor

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Further Comparison of Stochastic and Deterministic Models for the Optimal Control of Lanchester-Type Attrition Processes

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Submitted in partial fulfillment of the
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ABSTRACT

The optimal fire distribution policy obtained using a stochastic combat attrition model is compared with that for a deterministic one. The same optimal control problem for a homogeneous force in combat against a heterogeneous force of two homogeneous types is considered using two different models for the attrition mechanism in a fight-to-the-finish: the Lanchester-type differential equation formulation and its analagous stochastic version of a continuous parameter Markov chain with stationary transition probabilities. Considering dynamic programming methodology, a computer program was developed to numerically determine the optimal fire distribution policy (closed-loop or feedback) for the stochastic attrition process. Numerical values are generated for several parameter sets and compared with the optimal fire distribution policy for the corresponding deterministic attrition process. Results indicate that the optimal fire distribution policy for the stochastic model is more complex than the deterministic one.

TABLE OF CONTENTS

I. INTRODUCTION -- 8

  A. COMBAT MODELLING AS AN ANALYTIC TOOL --------- 8

  B. HISTORY OF THE LANCHESTER MODELS ------------- 9

II. PROBLEM STATEMENT -------------------------------- 11

  A. DETERMINISTIC ALLOCATION OF FIRE MODEL ------- 11

  B. STOCHASTIC ALLOCATION OF FIRE MODEL ---------- 13

    1. Formal Model ------------------------------ 13

    2. Underlying Poisson Process ---------------- 15

    3. Optimal Control/Dynamic Programming Problem ----------------------------------- 16

III. COMPUTER ANALYSIS ------------------------------ 22

  A. GENERAL -- 22

  B. PROGRAMMING ------------------------------------ 23

  C. INPUT PARAMETERS ------------------------------- 24

  D. ANALYSIS OF OUTPUT ----------------------------- 25

IV. CONCLUSIONS -------------------------------------- 31

APPENDIX A GENERAL CASE FOR ADMISSIBLE ORDER OF COMPUTING OPTIMAL EXPECTED VALUE FUNCTIONS ------- 32

APPENDIX B CATEGORIES OF COMPUTATIONS FOR OPTIMAL EXPECTED RETURN AND OPTIMAL ALLOCATION POLICY ---- 35

COMPUTER OUTPUT -------------------------------------- 37

COMPUTER PROGRAM ------------------------------------- 44

LIST OF REFERENCES ----------------------------------- 46

INITIAL DISTRIBUTION LIST ---------------------------- 47

LIST OF TABLES

Table

I. Input Parameters and Decision Factors ------------ 26
II. Allocation of Fire When Y Survivability Is Seriously Threatened ------------------------------ 30
III. Allocation of Fire When Y Survivability Is Almost Certain -- 30

LIST OF FIGURES

Figure

1. Allocation of Fire Between Forces ------------------ 11

2. System Dynamics of Stochastic Allocation of Fire Model -- 17

3. Parameter Input Interpretation --------------------- 28

I. INTRODUCTION

A. COMBAT MODELLING AS AN ANALYTICAL TOOL

Mathematical combat modelling is becoming increasingly important throughout the services as a tool for assistance in decision-making on new weapons systems and force levels and for gaining insights to their applications in specific military scenarios. It is of interest to continue research in this area to develop models which are realistic to a given combat situation and hence useful to the decision-maker and which are mathematically tractable. Current models in use employ either Monte Carlo simulation of combat as used in DYNTACS (Dynamic Tactical Simulator), or analytic methods as found in the Bonder IUA (Individual Unit Action) model. Exploration of the effects of various decision variables available to the combatants becomes extremely important when attempting to incorporate sound military tactics into any model under consideration. These decision variables could be explored heuristically or by applying optimization theory to each specific model. It is the intent of this thesis to examine an idealized question of optimal time sequential tactical decision-making through the application of optimization theory.

B. HISTORY OF THE LANCHESTER MODELS

The basis of the deterministic models used today are the equations formulated by Lanchester [6] in 1914 to describe the attrition process between two homogeneous combat forces: $\frac{dx}{dt} = -ay$ and $\frac{dy}{dt} = -bx$ where a, b are Lanchester attrition rates for the forces X and Y and x(t), y(t) are the force levels at time t. Koopman [7] formulated a stochastic extension to the purely deterministic model by incorporating state probabilities; i.e., the probability that at time t the force levels are x(t) and y(t). This extension will be referred to as the Lanchester stochastic process. This stochastic reformulation became a definite step forward in the theory of modelling, since it allows casualties to occur randomly over time.

Isbell and Marlow [2] were among the first to apply optimal time sequential decision-making to Lanchester models of combat. In particular, they developed a general fire programming problem for target allocation for combat among heterogeneous forces and employed techniques of Isaacs [1] in their solution. Later, Taylor [9] used a different approach (modern optimal control theory) to develop a more complete solution and presented a general solution algorithm for the basic problem. Specifically, the decision variable was the commander's allocation of fire in the case of combat between one homogeneous Y force against an X force of two homogeneous types in the deterministic Lanchester model formulation of the situation. Although this is the simplest

of all possible cases of this type, the general solution is quite complicated. Nonetheless, it does give useful insights to the problem of allocation of fire, and is a step forward in developing solutions for more complex situations.

Although the stochastic version of the Isbell-Marlow problem would theoretically be a better model since the random nature of combat attrition is considered, one might question whether or not similar results would be obtained; i.e. would the allocation of fire follow the same pattern as in the deterministic case? This question becomes highly relevent since to obtain a general analytic solution to the stochastic version is exceedingly complex and has only been done in a few specific cases. Powers [8] has made a comparison of these two versions for the prescribed duration battle and found that strategies were essentially the same as in the deterministic model, although in this case the comparison was awkward to make. This thesis will compare optimal strategies derived from the stochastic and deterministic versions of the Isbell-Marlow problem in the case of a fight to the finish. Hopefully, some insight will be gained into the basic structure of the stochastic model.

The next section of this thesis will review the deterministic optimization problem and the implications of its solution. Then a discussion of the stochastic version will follow with a presentation of the conditions for optimality. Finally, numerical results for specific cases of the stochastic version will be compared with the deterministic model.

II. PROBLEM STATEMENT

A. DETERMINISTIC ALLOCATION OF FIRE MODEL

The case under consideration[1] has the commander of a homogeneous Y force desiring to maximize the net worth of survivors in a battle between a heterogeneous X force of two homogeneous types (X_1 and X_2). A linear cardinal utility is imposed on the survivors as the objective function, and the decision variable $\phi(t)$ is the proportion of Y fire directed at the X_1 force at time t. The situation is diagrammed in Figure 1.

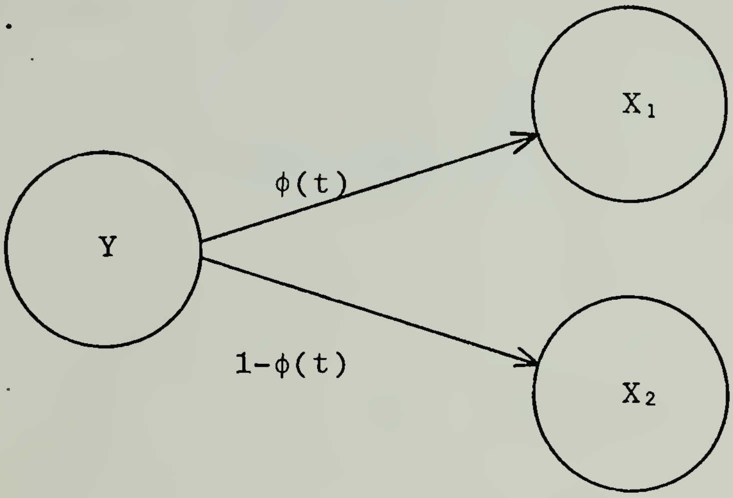

Figure 1. Allocation of Fire Between Forces

---

[1] A concise summary of the problem and its complete solution is given in Reference 10.

The optimal control problem for a fight to the finish is formally stated as

$$\underset{\phi(t)}{\text{Maximize}} \{ry(T)-px_1(T)-qx_2(T)\}$$

$$\text{Subject to:} \quad \frac{dx_1}{dt} = -\phi(t)a_1y$$

$$\frac{dx_2}{dt} = -\{1-\phi(t)\}a_2y$$

$$\frac{dy}{dt} = -b_1x_1 - b_2x_2 \qquad (1)$$

with constraints

$$x_1,\ x_2,\ y \geq 0$$

$$0 \leq t \leq T, \quad T \text{ unspecified}$$

$$0 \leq \phi(t) \leq 1$$

and initial conditions

$$x_1(0) = x_1^0, \quad x_2(0) = x_2^0, \quad y(0) = y_0$$

and stopping rule

either 1) $y(T) = 0$ (X force wins)

or 2) $x_1(T) = 0$ and $x_2(T) = 0$ (Y force wins)

Note that a_i is the attrition rate of the X_i force due to Y's fire, while b_i is the attrition rate of the Y force due to X_i's fire. Hence the factor a_ib_i can be interpreted as the rate of destruction of X_i's kill capability against Y. Also, since p and q are measures of the "worth" of the X_1 and X_2 survivors, a_1p is a measure of the rate of

destruction of an X_1 combatant's value to Y and a_2q the X_2 combatant's value to Y. These are the factors upon which the optimal fire distribution policy, call it ϕ^*, depends. It has been shown [9] that ϕ^* must be either zero or one except for at most one point in time.

Simply stated, there are two cases in which the Y force will always fire at X_1 until complete annihilation of the X_1 force then switch fire to X_2. If $x_1(T) = x_2(T) = 0$ and $a_1b_1>a_2b_2$, then $\phi^* = 1.0 \ \forall \ x_1>0$. Also, if $a_1b_1>a_2b_2$ and $a_1p\geq a_2q$, then $\phi^* = 1.0 \ \forall \ x_1>0$. In the first case if Y wins, he will always fire at the force whose kill capability he can destroy the fastest. In the second case, he will always fire at the force whose kill capability and survivor "worth" he can destroy faster. If the above conditions do not hold, the optimal fire distribution policy may shift during the course of battle, and the timing of this shift may be complex to describe.

B. STOCHASTIC ALLOCATION OF FIRE MODEL

1. Formal Model

If, in the above deterministic Lanchester-type formulation, the combat attrition process is considered to be Markovian, then the resulting stochastic optimal control problem may be stated as:

$$\underset{0\leq\phi\leq 1}{\text{Maximize}} \{E[rN(T)-pM_1(T)-qM_2(T)]\} \quad (2)$$

Subject to: random occurrence of casualties as a continuous parameter Markov Chain with transition probabilities corresponding to the deterministic Lanchester process (1)

with constraints

$N(t),\ M_1(t),\ M_2(t) \geq 0$

$0 \leq t \leq T$, T unspecified

$0 \leq \phi \leq 1$

and initial conditions

$P\{N(0)=n_0\}=1, \quad P\{M_1(0)=m_1^0\}=1, \quad P\{M_2(0)=m_2^0\}=1$

and stopping rule

either 1) $n(T)=0$ (X force wins)

or 2) $m_1(T)=0$ and $m_2(T)=0$ (Y force wins)

$N(t)$, $M_1(t)$, and $M_2(t)$ are random variables specifying force levels of the Y, X_1, and X_2 forces respectively, $E[\cdot]$ is mathematical expectation, and ϕ is again the proportion of Y firers directing their fire at the X_1 force at time t.

For the reader's convenience, some probabilistic aspects of Lanchester combat will be reviewed (see [3], [10] for more formal analysis) before the dynamic programming development of the fundamental functional equation for the stochastic control problem. Since this thesis is concerned with the fight to the finish (see stopping rule for (2) above), analysis by the author will then be presented which

logically connects this situation to the one developed by Powers and Taylor [8] for the prescribed duration battle.

2. Underlying Poisson Process

The occurrence of casualties forms a Poisson Process with the attrition rates associated with the rate of the process. Assuming independent and stationary increments and the probability of more than one casualty occurring in Δt to be negligible, the event probabilities are developed:

Define:

CX_i = Event of an X_i casualty in Δt; $i=1,2$

CY = Event of a Y casualty in Δt

C = Event of a casualty in Δt

and note that CX_1, CX_2, CY are mutually exclusive and they are collectively exhaustive of the event C.

Then:

$$P(CX_1) = \phi a_1 n \Delta t$$

$$P(CX_2) = (1-\phi) a_2 n \Delta t$$

$$P(CY) = (b_1 m_1 + b_2 m_2) \Delta t$$

$$P(C) = \phi a_1 n \Delta t + (1-\phi) a_2 n \Delta t + (b_1 m_1 + b_2 m_2) \Delta t$$

For the later dynamic programming problem to be developed it is necessary to have the probability of a casualty being from a particular force, conditioned on a casualty occurring. Hence, by Bayes rule:

$$P(CX_1|C) = \frac{a_1 n}{\phi a_1 n+(1-\phi)a_2 n+b_1 m_1+b_2 m_2}$$

$$P(CX_2|C) = \frac{(1-\phi)a_2 n}{\phi a_1 n+(1-\phi)a_2 n+b_1 m_1+b_2 m_2}$$

$$P(CY|C) = \frac{b_1 m_1+b_2 m_2}{\phi a_1 n+(1-\phi)a_2 n+b_1 m_1+b_2 m_2} \qquad (3)$$

3. Optimal Control/Dynamic Programming Problem

The problem as stated in equation (2) readily becomes a dynamic programming problem when the state of the system is taken as the three force levels at any arbitrary time t and a return function is associated with each possible force level. Define:

System State = $\{(m_1,m_2,n)|0\leq m_1\leq m_1^0, 0\leq m_2\leq m_2^0, 0\leq n\leq n_0\}$

Expected Return = $R(\phi;m_1,m_2,n) = E[rN(T)-pM_1(T)-qM_2(T)]$

Optimal Expected Return = $W(m_1,m_2,n) = \underset{\phi\varepsilon\Phi}{\text{Max}}\{R(\phi;m_1,m_2,n)\}$

Control Variable = $\phi \ \varepsilon \ \Phi = \{0,\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n},1\}$

Viewing the system in "backwards time", there are exactly three ways any particular state (m_1,m_2,n) can be realized through the random occurrence of casualties. The dynamic system is described diagrammatically in Figure 2.

The optimal expected return for any state (m_1,m_2,n) is thus the expected value of the optimal expected return from the three previous possible states when maximized over the control variable ϕ. Mathematically,

$$W(m_1,m_2,n) = \underset{\phi\varepsilon\Phi}{\text{Max}} \{W(m_1-1,m_2,n)P(CX_1/C)+W(m_1,m_2-1,n)P(CX_2/C)$$
$$+W(m_1,m_2,n-1)P(CY/C)\}.$$

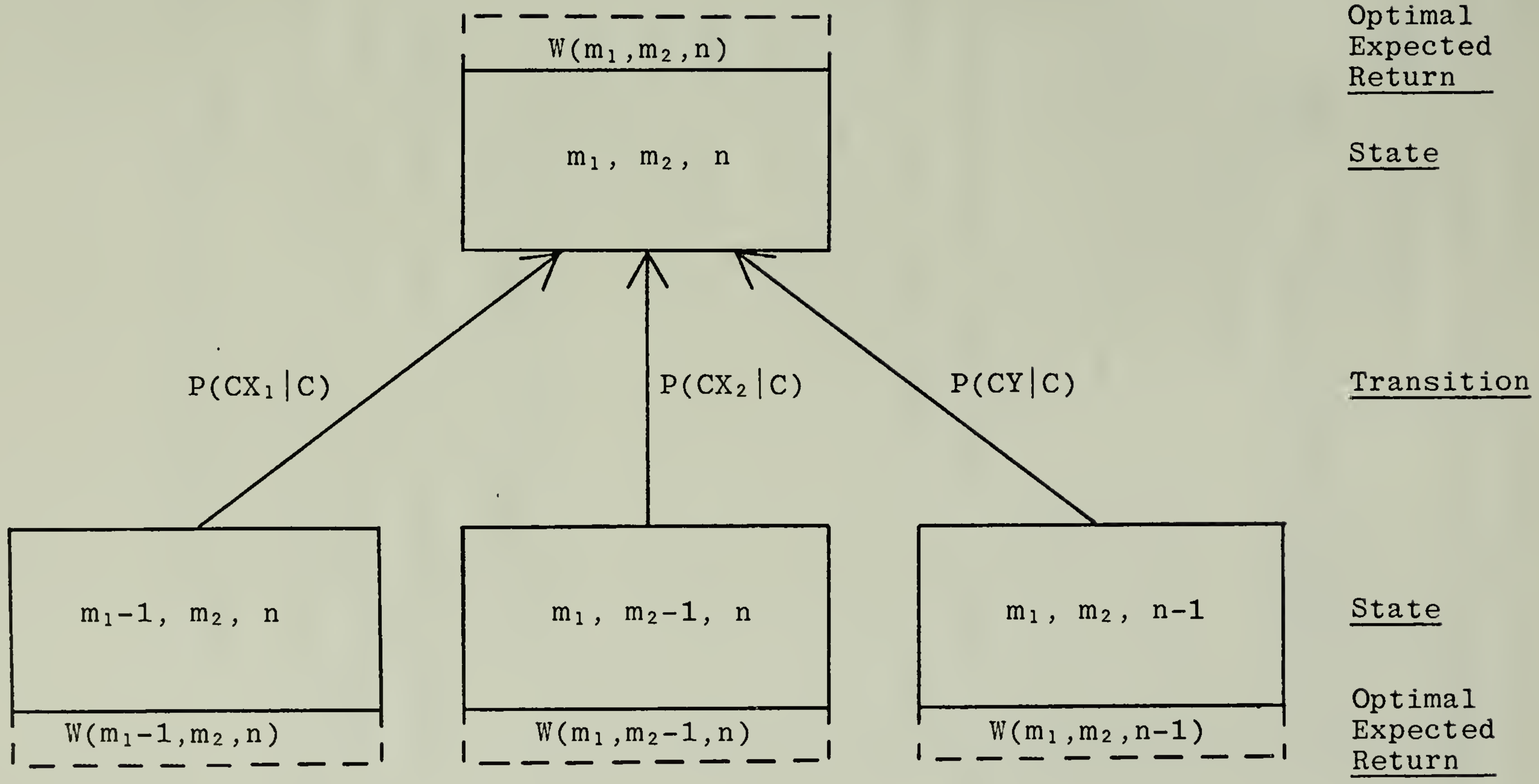

Figure 2. System Dynamics of Stochastic Allocation of Fire Model

Utilizing the probabilities previously developed (3), the fundamental functional equation for the stochastic allocation of fire model is obtained:

$$W(m_1,m_2,n) = \max_{\phi\varepsilon\Phi} \left\{\frac{\phi a_1 nW_1+(1-\phi)a_2 nW_2+(b_1 m_1+b_2 m_2)W_3}{\phi a_1 n+(1-\phi)a_2 n+b_1 m_1+b_2 m_2}\right\} \qquad (4)$$

The boundary conditions are:

1) $W(0,0,n) = rn \qquad n = 0,1,\ldots,n_0$

2) $W(m_1,m_2,0) = -pm_1-qm_2 \qquad m_1= 0,1,\ldots,m_1^0$

$m_2= 0,1,\ldots,m_2^0$

For notational simplicity, it is useful to define the following terms before developing the optimality conditions for the selection of ϕ:

$W = W(m_1,m_2,n)$

$W_1 = W(m_1-1,m_2,n)$

$W_2 = W(m_1,m_2-1,n)$

$W_3 = W(m_1,m_2,n-1)$

$$R(\phi) = \frac{\phi a_1 nW_1+(1-\phi)a_2 nW_2+(b_1 m_1+b_2 m_2)W_3}{\phi a_1 n+(1-\phi)a_2 n+b_1 m_1+b_2 m_2}$$

Rearranging terms of the objective function, a ratio of two linear functions of ϕ is obtained (momentarily assuming ϕ to be continuous over the closed interval [0,1]):

$$R(\phi) = \frac{\phi(a_1 nW_1-a_2 nW_2)+(b_1 m_1+b_2 m_2)W_3+a_2 nW_2}{\phi(a_1 n-a_2 n)+b_1 m_1+b_2 m_2+a_2 n}$$

Since $R(\phi)$ is such a ratio, it follows[2] that

$$R'(\phi) \begin{cases} >0 \Leftrightarrow & A>B \\ =0 \Leftrightarrow & A=B \\ <0 \Leftrightarrow & A<B \end{cases}$$

where

$$A = (a_1W_1-a_2W_2)(b_1m_1+b_2m_2+a_2n)$$

$$B = (a_1-a_2)[(b_1m_1+b_2m_2)W_3+a_2nW_2]$$

Thus, the optimality conditions for ϕ and optimal return for the stochastic allocation of fire model are

$$W = \underset{\phi\varepsilon\Phi}{\text{Max}} \{R(\phi)\} = \begin{cases} R(\phi=1) \Leftrightarrow & A>B \\ R(\phi') \quad \Leftrightarrow & A=B \qquad (\phi' \text{ arbitrary}) \\ R(\phi=0) \Leftrightarrow & A<B \end{cases} \tag{5}$$

There is no known general solution to this optimal control/dynamic programming problem. However, for any set of input parameters (a_1,a_2,b_1,b_2,p,q,r), a solution may be built up throughout the set of possible states. Optimal returns for the base cases may be explicitly found from the boundary conditions, and returns for succeeding states found in turn. However, these may only be computed in a particular order, since to find the optimal expected return for any

---

[2] $R(\phi) = \dfrac{a\phi+b}{c\phi+d} \Rightarrow R'(\phi) = \dfrac{ad-bc}{(c\phi+d)^2} \Rightarrow$

$$R'(\phi) \begin{cases} >0 \Leftrightarrow ad>bc \\ =0 \Leftrightarrow ad=bc \\ <0 \Leftrightarrow ad<bc \end{cases}$$

(m_1,m_2,n) state, the optimal expected returns must be known for the three previous states (m_1-1,m_2,n), (m_1,m_2-1,n), and $(m_1,m_2,n-1)$. (See Appendix A for the admissible order of states).

The fundamental equation (4) developed here may be obtained from the fundamental functional equation derived for the prescribed duration battle[3] by taking the maximum length of the battle to be very large; i.e. considering the steady state form of optimal return when the time derivative of the return vanishes:

$$0 = (b_1m_1+b_2m_2)[W_3-W]+ n\underset{\phi}{\text{Max}}\ [\phi a_1(W_1-W)+(1-\phi)a_2n(W_2-W)]$$

Placing all terms within the Max operator and rearranging yields

$$\underset{\phi}{\text{Max}}\{\phi a_1nW_1+(1-\phi)a_2nW_2+(b_1m_1+b_2m_2)W_3-[\phi a_1n+(1-\phi)a_2n$$
$$+b_1m_1+b_2m_2]W\} = 0$$

Factoring the coefficient of W gives

$$\underset{\phi}{\text{Max}}\{D(\phi)[\frac{\phi a_1n}{D(\phi)}W_1 + \frac{(1-\phi)a_2n}{D(\phi)}W_2 + \frac{(b_1m_1+b_2m_2)}{D(\phi)}W_3 - W]\} = 0$$

where $D(\phi) = \phi a_1n+(1-\phi)a_2n+b_1m_1+b_2m_2$

Since $D(\phi) > 0 \ \forall \phi, a_1, a_2, m_1, m_2, n$ under consideration it follows[4] that

---

[3] See p. 26-28 of Reference 8.

[4] $\underset{\phi}{\text{Max}}\{D(\phi)f(\phi)\}=0 \Leftrightarrow$ (1) $\forall \phi$, $D(\phi)f(\phi) \leq 0$ and
(2) $\exists \phi \ni: D(\phi)f(\phi)=0 \Leftrightarrow$ For $D(\phi) > 0$, (1) $\forall \phi, f(\phi) \leq 0$ and
(2) $\exists \phi \ni: f(\phi)=0 \Leftrightarrow \underset{\phi}{\text{Max}}\{f(\phi)\} = 0$

$$\max_{\phi}\left\{\frac{\phi a_1 n}{D(\phi)}W_1 + \frac{(1-\phi)a_2 n}{D(\phi)}W_2 + \frac{(b_1 m_1 + b_2 m_2)}{D(\phi)}W_3 - W\right\} = 0$$

$$\Rightarrow \quad W = \max_{\phi}\left\{\frac{\phi a_1 n}{D(\phi)}W_1 + \frac{(1-\phi)a_2 n}{D(\phi)}W_2 + \frac{(b_1 m_1 + b_2 m_2)}{D(\phi)}W_3\right\} .$$

This is readily recognizable as the fundamental equation (4) for the fight to the finish.

III. COMPUTER ANALYSIS

A. GENERAL

Output information required for analyzing the stochastic model consists of three elements: the state of the system, its associated optimal expected return, and its optimal fire distribution policy, tabulated over all possible states for a given initial force level. With this information, a direct comparison with the deterministic model output of the allocation policy and return for each state is made possible. The comparison can be made easily by inspection whenever $a_1b_1 > a_2b_2$ and $a_1p \geq a_2q$, since the deterministic model solution is always to fire at X_1 ($\phi^*=1$) when $x_1 > 0$. Also, when $a_1b_1 > a_2b_2$ and $a_1p < a_2q$, the deterministic solution remains the same if Y wins. Conditions that guarantee that Y wins are readily determined for the deterministic problem, but in the stochastic model whether or not Y wins is probabilistic. Thus Y's decision may be significantly state dependent and he must continually evaluate his policy at each stage by comparing the tradeoff between his rate of destruction of each X force's kill capability and his value of the "worth" of each X force to him as well as the value he places on his own forces. The underlying optimal fire distribution policy is tractable in this case, but highly complex.

B. PROGRAMMING

For possible future applications, the computer program is listed at the end of this thesis. Since the return for any possible state is a function of the returns for the three previous possible states, the structural key is the ordering of admissible states. Hence this became the basic program with the optimal return and allocation policy for each state being computed in a subroutine.

Several manual checks were made to verify the program output. Admissible ordering of states previously tabulated[5] proved to be in error; Appendix A is the corrected ordering for the general case. This ordering has been verified by manual tabulation of all possible states through several different force levels. Output from the algorithm itself was also manually verified through random selection of examples from five specific force level categories where W and ϕ^* are computed by different methods. Only one category involves the computation of W and ϕ^* by the method presented in section IIB3; in all other cases ϕ^* is trivially zero or one and W is easily computed once this known. Appendix B lists the five categories and computations used by the computer. Printing of output is restricted only to those states with positive components due to the trivial allocation policies of those with zero components. The initial force levels for X_1, X_2, and Y are required for program

---

[5] See p. 36 in Reference 8.

set-up. Given a desired initial force level of m_1^0, m_2^0, and n_0, the expected return array must be dimensioned $(m_1^0+1) \times (m_2^0+1) \times (n_0+1)$ and the upper value of the second DO-Loop must be n_0. The program is designed for battles where $m_1^0=m_2^0=n_0$.

C. INPUT PARAMETERS

Input required for the program consists of values for a_1, $a_2, b_1, b_2, p, q, r, m_1^0, m_2^0$, and n_0. Nineteen runs were made with various values for these parameters to allow comparison with the deterministic solution in Reference 9. The values were selected in order to represent the several tradeoffs upon which the Y force commander must make a judgment before allocating fire. Major factors influencing his decision in the deterministic model are:

1) Rate of destruction of X_1 kill capability versus X_2 kill capability, measured by the ratio $R = \frac{a_1 b_1}{a_2 b_2}$
2) Rate of destruction of X_1 "worth" to Y versus X_2 "worth to Y, measured by the ratio $\delta = \frac{a_1 p}{a_2 q}$.

Output from the stochastic model shows that, in addition to the above, the Y commander should also take into account how dangerous the X forces are relative to him (measured by b_1 and b_2) and how much he values his survivors (measured by r).

The system was analyzed using initial force levels of ten for the X_1, X_2, and Y forces. In all cases, R is greater than one; R less than one would give a "mirror image" result. The parameter δ fluctuates about one in varying degrees and

r ranges from one to one hundred thousand. The parameter sets and major decision factors are listed in Table I. Sets discussed in the next section are outlined in the table, and portions of each set's output is listed after Appendix B.

D. ANALYSIS OF OUTPUT

The major purpose of this thesis was to provide a computer program for solving the stochastic allocation model to give a means of comparison with the deterministic model. Significant implications were obtained by selecting parameter sets 1,7,8,9,10,11, and 12 as a basis for comparison. In each of these sets $R>1$ and $\delta>1$, so that the deterministic solution is always $\phi^*=1.0$. The output from the stochastic model has shown that, even when the two conditions above hold, the Y commander should consider several other variables and their interactions before deciding upon which force to direct his fire. It is emphasized that the results discussed below are based upon a logical intuition derived from a physical interpretation of the various input parameters.

The optimal allocation policy $\phi^*=1.0$ was obtained for all force levels with the input of parameter set one. Note that Y is twice as effective against the X_1 force as he is against the X_2 force, and that a survivor's "worth" for each force is weighted the same. This result also follows in the deterministic model.

Parameter sets 7,8,9,10,11, and 12 utilize, between each set, the same attrition rate coefficients and the same value

Data	Parameters							Deterministic Decision Factors	
Set No.	a_1	a_2	b_1	b_2	p	q	r	R	δ
1.	2	1	1	1	1	1	1	2.0	2.0
2.	2	1	1	1	1	5	100	2.0	0.4
3.	2	1	1	1	1	10	1	2.0	0.2
4.	2	1	1	1	1	10	10	2.0	0.2
5.	2	1	1	1	1	10	100	2.0	0.2
6.	2	1	1	1	1	100	1	2.0	0.02
7.	11	1	1	10	1	1	1	1.1	11.0
8.	11	1	1	10	1	1	10	1.1	11.0
9.	11	1	1	10	1	1	100	1.1	11.0
10.	11	1	1	10	1	1	1000	1.1	11.0
11.	11	1	1	10	1	1	10000	1.1	11.0
12.	11	1	1	10	1	1	100000	1.1	11.0
13.	11	1	1	10	1	10	1	1.1	1.1
14.	11	1	1	10	1	20	1	1.1	0.55
15.	11	1	1	10	1	20	10	1.1	0.55
16.	11	1	1	10	1	20	100	1.1	0.55
17.	11	1	1	10	1	100	1	1.1	0.11
18.	11	1	1	10	1	100	10	1.1	0.11
19.	11	1	1	10	1	100	100	1.1	0.11

Table I. Input Parameters and Decision Factors

placed upon the X_1 and X_2 survivors. The only change occurring is the value Y placed upon his own survivors, which increases by a factor of ten through each set. The attrition rate coefficients were selected to represent two different concepts. When viewed individually, the coefficients represent a marked difference in effectiveness of individual fire, but when viewed as products $(a_i b_i)$ they reflect almost equal $(R=1.1)$ rates of destruction of kill capability. Tactically considering the Y force as being attacked by two X forces, Figure 3 represents the situation as viewed by the Y commander with his interpretation of the factors involved.

Constant throughout this situation are the facts that Y can destroy the X_2 kill capability almost as fast as X_1's, while an X_2 element is individually much more dangerous to Y than an X_1 element. The results tend to show that, as long as the Y force has a "good" chance surviving, his fire will be directed against the X_1 force which he can destroy faster. However, when his survivability is threatened, fire will be shifted to the more dangerous X_2 force. Finally, it appears that when destruction of the Y force is inevitable (Y has a very low force level compared to X), Y will again shift fire to the X_1 force to destroy as many of the enemy as he can before he himself is destroyed. Throughout this situation, as Y places more value upon his own survivors (an increase in r), he will direct fire more frequently to the more dangerous X force. Reinforcing this is the observation that for these selected parameter sets, $\forall r' < r$,

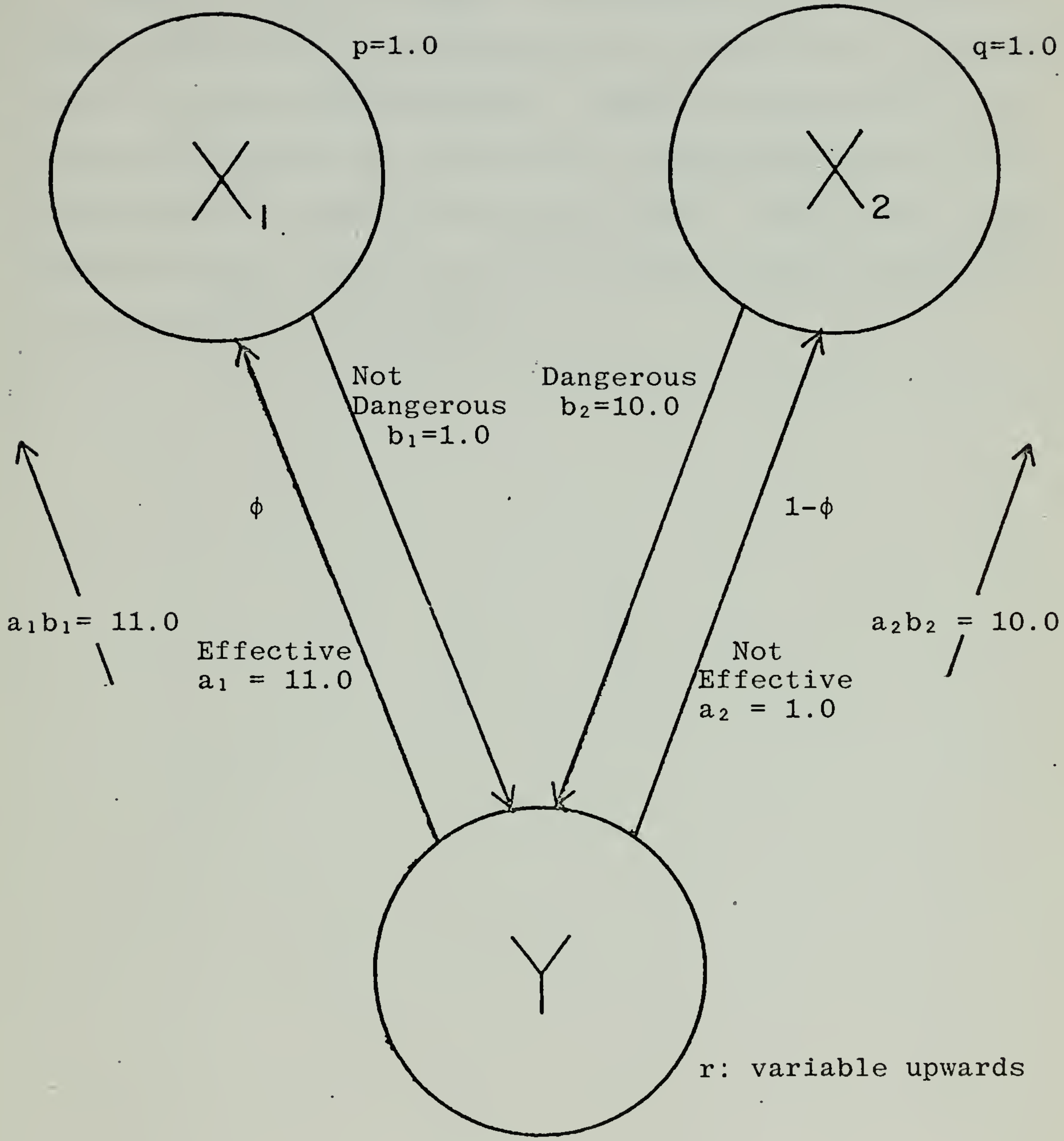

Figure 3. Parameter Input Interpretation

$\phi^*=1.0$ at $r \Rightarrow \phi^*=1.0$ at r' for a given force level, but the reverse implication does not hold.

Tables II and III depict the phenomena discussed above. Within each table, the probability of Y surviving is lowered as the Y force level decreases. Table entries are ϕ^*, the optimal allocation of fire for the corresponding force levels and value Y places on his survivors. Figures were taken from the output in data sets 7,8,9,10,11, and 12 going from left to right in the tables.

Force Level			Value of Y Survivor = r					
m_1	m_2	n	1	10	100	1,000	10,000	100,000
10	2	5	1	1	0	0	0	0
10	2	4	1	1	0	0	0	0
10	2	3	1	1	0	0	0	0
10	2	2	1	1	1	0	0	0
10	2	1	1	1	1	1	1	0

Table II

Allocation of Fire When Y Survivability Is Seriously Threatened.

Force Level			Value of Y Survivor = r					
m_1	m_2	n	1	10	100	1,000	10,000	100,000
1	1	10	1	1	1	1	1	1
1	1	9	1	1	1	1	1	1
1	1	8	1	1	1	1	1	1
1	1	7	1	1	1	1	1	1
1	1	6	1	1	0	0	0	0
1	1	5	1	0	0	0	0	0

Table III

Allocation of Fire When Y Survivability Is Almost Certain.

IV. CONCLUSIONS

The purpose of this thesis was to make a comparison between an optimal time-sequential fire distribution policy based upon a deterministic attrition mechanism and one based upon a stochastic process. This was done for the simplest possible fire distribution problem: a homogeneous Y force in Lanchester combat against a heterogeneous X force of two types. The optimal time-sequential fire distribution policy was determined for both the deterministic and stochastic optimal control problems, and the results were compared. The specific comparison made indicated that even in the simplest case, the deterministic model nowhere reflects the many factors considered in the stochastic model in determining the optimal allocation of fire. Even when the deterministic model tends to be simplistic, the stochastic version is apparently highly complex with interactions amony many variables determining the optimal solution. Future work remains to be done on determining the true functional dependence of the optimal fire distribution policy for a stochastic attrition mechanism.

APPENDIX A

GENERAL CASE FOR ADMISSIBLE ORDER OF COMPUTING OPTIMAL EXPECTED VALUE FUNCTIONS

The ordering is listed for the condition that $m_1^0=m_2^0=n^0$. The entire sequence must be followed for each i in the succession i=1,2,...,n. Numbers to the left of a section identify the corresponding DO-Loop in the main computer program.

	m_1	m_2	n
	i	0	0
	0	i	0
	0	0	i
	i	1	0
200	i	2	0
	⋮	⋮	⋮
	i	i-1	0
	1	i	0
	2	i	0
300	⋮	⋮	⋮
	i-1	i	0
	i	i	0
	0	i	1
	0	i	2
400	⋮	⋮	⋮
	0	i	i-1

	m_1	m_2	n
	0	1	i
	0	2	i
500	⋮	⋮	⋮
	0	i-1	i
	0	i	i
	i	0	1
	i	0	2
600	⋮	⋮	⋮
	i	0	i-1
	1	0	i
	2	0	i
700	⋮	⋮	⋮
	i-1	0	i
	i	0	i

	m_1	m_2	n
	i	1	1
	i	2	1
	⋮	⋮	⋮
	i	i-1	1
	i	1	2
	i	2	2
	⋮	⋮	⋮
900	i	i-1	2
	⋮	⋮	⋮
	i	1	i-1
	i	2	i-1
	⋮	⋮	⋮
	i	i-1	i-1
	1	i	1
	2	i	1
	⋮	⋮	⋮
	i	i	1
	1	i	2
	2	i	2
	⋮	⋮	⋮
1200	i	i	2
	⋮	⋮	⋮
	1	i	i-2
	2	i	i-2
	⋮	⋮	⋮
	i	i	i-2
	1	i	i-1
	2	i	i-1
	⋮	⋮	⋮
	i-1	i	i-1
	1	1	i
	2	1	i
	⋮	⋮	⋮
	i	1	i
	1	2	i
	2	2	i
	⋮	⋮	⋮
	i	2	i
1500	⋮	⋮	⋮
	1	i-2	i
	2	i-2	i
	⋮	⋮	⋮
	i	i-2	i
	1	i-1	i
	1	i	i
	2	i-1	i
	2	i	i
	⋮	⋮	⋮
	i-2	i-1	i
	i-2	i	i
	i-1	i-1	i

m_1	m_2	n
i	i	i-1
i-1	i	i
i	i-1	i
i	i	i

APPENDIX B

CATEGORIES OF COMPUTATIONS FOR OPTIMAL EXPECTED RETURN AND OPTIMAL ALLOCATION POLICY

Boundary Conditions

1) $W(0,0,n) = rn \qquad n=1,\dots,n_0$

2) $W(m_1,m_2,0) = -pm_1-qm_2 \qquad m_1=0,1,\dots,m_1^0$

$m_2=0,1,\dots,m_2^0$

Special Cases

1) $m_1=0,\ m_2>0,\ n>0$

$$W(0,m_2,n)=\frac{b_2m_2W(0,m_2,n-1)+a_2nW(0,m_2-1,n)}{b_2m_2+a_2n}$$

$\phi^*=0.0$

2) $m_1>0,\ m_2=0,\ n>0$

$$W(m_1,0,n)=\frac{b_1m_1W(m_1,0,n-1)+a_1nW(m_1-1,0,n)}{b_1m_1+a_1n}$$

$\phi^*=1.0$

General Case

1) $m_1>0,\ m_2>0,\ n>0$

Let $A = \{a_1W(m_1-1,m_2,n)-a_2W(m_1,m_2-1,n)\}(b_1m_1+b_2m_2+a_2n)$

$B = (a_1-a_2)\{(b_1m_1+b_2m_2)W(m_1,m_2,n-1)+a_3nW(m_1,m_2-1,n)\}$

Then

$A > B \Rightarrow \phi^* = 1.0$

$A < B \Rightarrow \phi^* = 0.0$

And

$$W(m_1,m_2,n) = [\phi^* a_1 n W(m_1-1,m_2,n)+(1-\phi^*)a_2 n W(m_1,m_2-1,n) +(b_1 m_1+b_2 m_2)W(m_1,m_2,n-1)] \;/\; [\phi^* a_1 n+(1-\phi^*)a_2 n +b_1 m_1+b_2 m_2]$$

```
1               A1=  2.000        B1=  1.000
                A2=  1.000        B2=  1.000
                 P=  1.000         Q=  1.000           R=          1.0


                  STATES                 EXPECTED                PHI
             M1    M2      N              RETURN                 STAR


              1     1      1               -1.00                 1.00
              2     1      1               -2.20                 1.00
              1     2      1               -2.33                 1.00
              1     1      2                0.56                 1.00
              2     2      1               -3.44                 1.00
              1     2      2               -1.00                 1.00
              2     1      2               -0.63                 1.00
              2     2      2               -2.22                 1.00
              3     1      1               -3.40                 1.00
              3     2      1               -4.56                 1.00
              3     1      2               -2.01                 1.00
              3     2      2               -3.52                 1.00
              1     3      1               -3.53                 1.00
              2     3      1               -4.58                 1.00
              3     3      1               -5.64                 1.00
              1     3      2               -2.54                 1.00
              2     3      2               -3.67                 1.00
              1     1      3                2.08                 1.00
              2     1      3                1.18                 1.00
              3     1      3               -0.10                 1.00
              1     2      3                0.70                 1.00
              1     3      3               -1.02                 1.00
              2     2      3               -0.47                 1.00
              3     3      2               -4.86                 1.00
              2     3      3               -2.22                 1.00
              3     2      3               -1.86                 1.00
              3     3      3               -3.54                 1.00
              4     1      1               -4.54                 1.00
              4     2      1               -5.64                 1.00
              4     3      1               -6.70                 1.00
              4     1      2               -3.42                 1.00
              4     2      2               -4.79                 1.00
              4     3      2               -6.03                 1.00
              4     1      3               -1.61                 1.00
              4     2      3               -3.32                 1.00
              4     3      3               -4.88                 1.00
              1     4      1               -4.63                 1.00
              2     4      1               -5.66                 1.00
              3     4      1               -6.70                 1.00
              4     4      1               -7.74                 1.00
              1     4      2               -3.90                 1.00
              2     4      2               -4.96                 1.00
              3     4      2               -6.07                 1.00
              4     4      2               -7.18                 1.00
              1     4      3               -2.71                 1.00
              2     4      3               -3.83                 1.00
              3     4      3               -5.04                 1.00
              1     1      4                3.39                 1.00
              2     1      4                2.79                 1.00
              3     1      4                1.82                 1.00
              4     1      4                0.50                 1.00
              1     2      4                2.37                 1.00
              2     2      4                1.42                 1.00
              3     2      4                0.16                 1.00
              4     2      4               -1.33                 1.00
              1     3      4                0.80                 1.00
              1     4      4               -1.04                 1.00
              2     3      4               -0.36                 1.00
              2     4      4               -2.24                 1.00
```

```
1          A1= 11.000        B1=  1.000
           A2=  1.000        B2= 10.000
            P=  1.000         Q=  1.000         R=          1.0


                STATES                  EXPECTED              PHI
          M1     M2      N               RETURN               STAR

           1      1      1               -1.41                1.00
           2      1      1               -2.24                1.00
           1      2      1               -2.64                1.00
           1      1      2               -0.70                1.00
           2      2      1               -3.55                1.00
           1      2      2               -2.21                1.00
           2      1      2               -1.24                1.00
           2      2      2               -2.88                1.00
           3      1      1               -3.19                1.00
           3      2      1               -4.53                1.00
           3      1      2               -1.97                1.00
           3      2      2               -3.72                1.00
           1      3      1               -3.73                1.00
           2      3      1               -4.67                1.00
           3      3      1               -5.67                1.00
           1      3      2               -3.38                1.00
           2      3      2               -4.15                1.00
           1      1      3                0.14                1.00
           2      1      3               -0.23                1.00
           3      1      3               -0.72                1.00
           1      2      3               -1.78                1.00
           1      3      3               -3.06                1.00
           2      2      3               -2.22                1.00
           3      3      2               -5.06                1.00
           2      3      3               -3.60                1.00
           3      2      3               -2.84                1.00
           3      3      3               -4.33                1.00
           4      1      1               -4.20                1.00
           4      2      1               -5.54                1.00
           4      3      1               -6.67                1.00
           4      1      2               -2.84                1.00
           4      2      2               -4.67                1.00
           4      3      2               -6.04                1.00
           4      1      3               -1.35                1.00
           4      2      3               -3.61                1.00
           4      3      3               -5.20                1.00
           1      4      1               -4.78                1.00
           2      4      1               -5.75                1.00
           3      4      1               -6.74                1.00
           4      4      1               -7.75                1.00
           1      4      2               -4.48                1.00
           2      4      2               -5.31                1.00
           3      4      2               -6.26                1.00
           4      4      2               -7.25                1.00
           1      4      3               -4.20                1.00
           2      4      3               -4.82                1.00
           3      4      3               -5.64                1.00
           1      1      4                1.19                1.00
           2      1      4                0.88                1.00
           3      1      4                0.52                1.00
           4      1      4                0.07                1.00
           1      2      4               -1.26                1.00
           2      2      4               -1.58                1.00
           3      2      4               -2.01                1.00
           4      2      4               -2.58                1.00
           1      3      4               -2.77                1.00
           1      4      4               -3.95                1.00
           2      3      4               -3.12                1.00
           2      4      4               -4.38                1.00
```

```
1            A1=  11.000       B1=   1.000
             A2=   1.000       B2=  10.000
              P=   1.000        Q=   1.000          R=          10.0


                  STATES               EXPECTED               PHI
            M1     M2      N            RETURN                STAR

             1      1      1             -1.00               1.00
             2      1      1             -2.04               1.00
             1      2      1             -2.62               1.00
             1      1      2              2.16               0.0
             2      2      1             -3.54               1.00
             1      2      2             -2.01               1.00
             2      1      2              0.89               0.0
             2      2      2             -2.78               1.00
             3      1      1             -3.10               1.00
             3      2      1             -4.53               1.00
             3      1      2             -0.43               0.0
             3      2      2             -3.67               1.00
             1      3      1             -3.73               1.00
             2      3      1             -4.67               1.00
             3      3      1             -5.67               1.00
             1      3      2             -3.37               1.00
             2      3      2             -4.14               1.00
             1      1      3              8.06               0.0
             2      1      3              6.53               0.0
             3      1      3              4.90               0.0
             1      2      3             -0.75               0.0
             1      3      3             -2.98               1.00
             2      2      3             -1.56               1.00
             3      3      2             -5.06               1.00
             2      3      3             -3.55               1.00
             3      2      3             -2.43               1.00
             3      3      3             -4.31               1.00
             4      1      1             -4.17               1.00
             4      2      1             -5.54               1.00
             4      3      1             -6.67               1.00
             4      1      2             -1.80               0.0
             4      2      2             -4.65               1.00
             4      3      2             -6.04               1.00
             4      1      3              3.21               0.0
             4      2      3             -3.36               1.00
             4      3      3             -5.19               1.00
             1      4      1             -4.78               1.00
             2      4      1             -5.75               1.00
             3      4      1             -6.74               1.00
             4      4      1             -7.75               1.00
             1      4      2             -4.48               1.00
             2      4      2             -5.31               1.00
             3      4      2             -6.26               1.00
             4      4      2             -7.25               1.00
             1      4      3             -4.19               1.00
             2      4      3             -4.82               1.00
             3      4      3             -5.63               1.00
             1      1      4             16.51               0.0
             2      1      4             14.72               0.0
             3      1      4             12.83               0.0
             4      1      4             10.85               0.0
             1      2      4              2.01               0.0
             2      2      4              0.94               0.0
             3      2      4             -0.17               0.0
             4      2      4             -1.30               1.00
             1      3      4             -2.37               1.00
             1      4      4             -3.92               1.00
             2      3      4             -2.87               1.00
             2      4      4             -4.36               1.00
```

```
1            A1= 11.000         B1=  1.000
             A2=  1.000         B2= 10.000
              P=  1.000          Q=  1.000          R=        100.0


                  STATES                  EXPECTED              PHI
             M1     M2      N              RETURN               STAR

             1      1       1                 5.80              0.0
             2      1       1                 3.17              0.0
             1      2       1                -2.49              1.00
             1      1       2                34.95              0.0
             2      2       1                -3.50              1.00
             1      2       2                 0.77              0.0
             2      1       2                29.21              0.0
             2      2       2                -0.77              0.0
             3      1       1                 0.57              0.0
             3      2       1                -4.51              1.00
             3      1       2                23.21              0.0
             3      2       2                -2.30              0.0
             1      3       1                -3.73              1.00
             2      3       1                -4.67              1.00
             3      3       1                -5.67              1.00
             1      3       2                -3.27              1.00
             2      3       2                -4.10              1.00
             1      1       3                91.09              0.0
             2      1       3                81.48              0.0
             3      1       3                71.46              0.0
             1      2       3                12.06              0.0
             1      3       3                -1.92              0.0
             2      2       3                 9.10              0.0
             3      3       2                -5.04              1.00
             2      3       3                -2.97              0.0
             3      2       3                 6.22              0.0
             3      3       3                -4.01              1.00
             4      1       1                -1.80              0.0
             4      2       1                -5.53              1.00
             4      3       1                -6.67              1.00
             4      1       2                17.27              0.0
             4      2       2                -3.78              0.0
             4      3       2                -6.03              1.00
             4      1       3                61.25              0.0
             4      2       3                 3.45              0.0
             4      3       3                -5.03              1.00
             1      4       1                -4.78              1.00
             2      4       1                -5.75              1.00
             3      4       1                -6.74              1.00
             4      4       1                -7.75              1.00
             1      4       2                -4.48              1.00
             2      4       2                -5.31              1.00
             3      4       2                -6.26              1.00
             4      4       2                -7.25              1.00
             1      4       3                -4.14              1.00
             2      4       3                -4.79              1.00
             3      4       3                -5.62              1.00
             1      1       4               172.85              0.0
             2      1       4               159.37              0.0
             3      1       4               145.44              0.0
             4      1       4               131.18              0.0
             1      2       4                37.78              0.0
             2      2       4                32.22              0.0
             3      2       4                26.84              0.0
             4      2       4                21.69              0.0
             1      3       4                 2.62              0.0
             1      4       4                -3.53              0.0
             2      3       4                 0.94              0.0
             2      4       4                -4.15              1.00
```

```
1              A1= 11.000       B1=  1.000
               A2=  1.000       B2= 10.000
                P=  1.000        Q=  1.000          R=       1000.0


                  STATES               EXPECTED             PHI
              M1    M2     N            RETURN              STAR

               1     1     1              74.55             0.0
               2     1     1              56.87             0.0
               1     2     1               0.52             0.0
               1     1     2             363.52             0.0
               2     2     1              -1.35             0.0
               1     2     2              32.09             0.0
               2     1     2             313.71             0.0
               2     2     2              24.90             0.0
               3     1     1              39.75             0.0
               3     2     1              -3.14             0.0
               3     1     2             261.81             0.0
               3     2     2              18.06             0.0
               1     3     1              -3.69             1.00
               2     3     1              -4.67             1.00
               3     3     1              -5.67             1.00
               1     3     2              -1.52             0.0
               2     3     2              -2.93             0.0
               1     1     3             921.88             0.0
               2     1     3             832.07             0.0
               3     1     3             738.75             0.0
               1     2     3             143.31             0.0
               1     3     3              11.26             0.0
               2     2     3             121.76             0.0
               3     3     2              -4.31             0.0
               2     3     3               7.76             0.0
               3     2     3             101.22             0.0
               3     3     3               4.48             0.0
               4     1     1              25.01             0.0
               4     2     1              -4.76             0.0
               4     3     1              -6.67             1.00
               4     1     2             210.80             0.0
               4     2     2              11.82             0.0
               4     3     2              -5.65             0.0
               4     1     3             644.00             0.0
               4     2     3              82.06             0.0
               4     3     3               1.47             0.0
               1     4     1              -4.78             1.00
               2     4     1              -5.75             1.00
               3     4     1              -6.74             1.00
               4     4     1              -7.75             1.00
               1     4     2              -4.44             1.00
               2     4     2              -5.30             1.00
               3     4     2              -6.25             1.00
               4     4     2              -7.25             1.00
               1     4     3              -3.37             0.0
               2     4     3              -4.43             0.0
               3     4     3              -5.46             1.00
               1     1     4            1736.61             0.0
               2     1     4            1606.68             0.0
               3     1     4            1472.85             0.0
               4     1     4            1336.28             0.0
               1     2     4             398.24             0.0
               2     2     4             350.21             0.0
```

```
1         A1= 11.000    B1=  1.000
          A2=  1.000    B2= 10.000
           P=  1.000     Q=  1.000     R=     10000.0


             STATES             EXPECTED           PHI
         M1   M2   N             RETURN            STAR

          1    1    1          762.05          0.0
          2    1    1          593.85          0.0
          1    2    1           31.77          0.0
          1    1    2         3649.27          0.0
          2    2    1           21.99          0.0
          1    2    2          346.34          0.0
          2    1    2         3158.71          0.0
          2    2    2          283.39          0.0
          3    1    1          431.53          0.0
          3    2    1           13.19          0.0
          3    1    2         2647.77          0.0
          3    2    2          223.95          0.0
          1    3    1           -2.88          0.0
          2    3    1           -4.18          0.0
          3    3    1           -5.44          0.0
          1    3    2           18.28          0.0
          2    3    2           12.73          0.0
          1    1    3         9229.86          0.0
          2    1    3         8338.01          0.0
          3    1    3         7411.63          0.0
          1    2    3         1456.78          0.0
          1    3    3          145.21          0.0
          2    2    3         1249.94          0.0
          3    3    2            7.67          0.0
          2    3    3          118.78          0.0
          3    2    3         1053.30          0.0
          3    3    3           94.81          0.0
          4    1    1          293.16          0.0
          4    2    1            5.97          0.0
          4    3    1           -6.62          1.00
          4    1    2         2146.15          0.0
          4    2    2          170.60          0.0
          4    3    2            3.23          0.0
          4    1    3         6471.47          0.0
          4    2    3          870.69          0.0
          4    3    3           73.56          0.0
          1    4    1           -4.78          1.00
          2    4    1           -5.75          1.00
          3    4    1           -6.74          1.00
          4    4    1           -7.75          1.00
          1    4    2           -3.70          0.0
          2    4    2           -4.91          0.0
          3    4    2           -6.10          0.0
          4    4    2           -7.20          1.00
          1    4    3            6.45          0.0
          2    4    3            3.34          0.0
          3    4    3            0.48          0.0
          1    1    4        17374.14          0.0
          2    1    4        16079.75          0.0
          3    1    4        14746.99          0.0
          4    1    4        13387.29          0.0
          1    2    4         4003.55          0.0
          2    2    4         3531.45          0.0
```

```
1          A1= 11.000        B1=  1.000
           A2=  1.000        B2= 10.000
            P=  1.000         Q=  1.000          R=   100000.0


          STATES                EXPECTED              PHI
    M1      M2       N           RETURN               STAR

     1       1       1           7637.05              0.0
     2       1       1           5963.67              0.0
     1       2       1            344.27              0.0
     1       1       2          36506.70              0.0
     2       2       1            255.46              0.0
     1       2       2           3488.83              0.0
     2       1       2          31608.75              0.0
     2       2       2           2868.24              0.0
     3       1       1           4349.30              0.0
     3       2       1            176.43              0.0
     3       1       2          26507.36              0.0
     3       2       2           2282.90              0.0
     1       3       1              6.88              0.0
     2       3       1              2.89              0.0
     3       3       1             -0.63              0.0
     1       3       2            217.91              0.0
     2       3       2            171.44              0.0
     1       1       3          92309.56              0.0
     2       1       3          83397.38              0.0
     3       1       3          74140.44              0.0
     1       2       3          14591.42              0.0
     1       3       3           1486.16              0.0
     2       2       3          12531.73              0.0
     3       3       2            129.85              0.0
     2       3       3           1230.90              0.0
     3       2       3          10574.16              0.0
     3       3       3           1000.21              0.0
     4       1       1           2974.66              0.0
     4       2       1            113.23              0.0
     4       3       1             -3.56              0.0
     4       1       2          21499.57              0.0
     4       2       2           1758.33              0.0
     4       3       2             94.32              0.0
     4       1       3          64746.18              0.0
     4       2       3           8756.98              0.0
     4       3       3            796.70              0.0
     1       4       1             -4.71              1.00
     2       4       1             -5.73              1.00
     3       4       1             -6.74              1.00
     4       4       1             -7.75              1.00
     1       4       2              5.64              0.0
     2       4       2              2.32              0.0
     3       4       2             -0.67              0.0
     4       4       2             -3.31              0.0
     1       4       3            106.59              0.0
     2       4       3             84.23              0.0
     3       4       3             64.60              0.0
     1       1       4         173749.38              0.0
     2       1       4         160810.50              0.0
     3       1       4         147488.19              0.0
     4       1       4         133897.25              0.0
     1       2       4          40056.69              0.0
     2       2       4          35343.85              0.0
```

OPTIMAL ALLOCATION OF FIRE FOR A STOCHASTIC LANCHESTER-TYPE ATTRITION PROCESS

```
   10 FORMAT (10X,3I6)
   50 FORMAT (6F8.4,F10.4)
   75 FORMAT ('1',10X,'A1=',F7.3,5X,'B1=',F7.3,/,11X,'A2=',
     1F7.3,5X,'B2=',F7.3,/,12X,'P=',F7.3,6X,'Q=',F7.3,7X,
     2'R=',F10.1,////)
  150 FORMAT (14X,'STATES',10X,'EXPECTED',12X,'PHI')
  250 FORMAT (10X,'M1',3X,'M2',4X,'N',9X,'RETURN',13X,
     1'STAR',//)
  350 FORMAT ('1')
      DIMENSION W(11,11,11)
      COMMON A1,A2,B1,B2,P,Q,R,W
      DO 1700 INDEX=1,500
      READ (5,50) A1,A2,B1,B2,P,Q,R
      IF (A1.EQ.999.0) GO TO 1800
      WRITE (6,75) A1,B1,A2,B2,P,Q,R
      WRITE (6,150)
      WRITE (6,250)
      NMAX=10
      CALL COMP (0,0,0)
      CALL COMP (1,0,0)
      CALL COMP (0,1,0)
      CALL COMP (0,0,1)
      CALL COMP (1,1,0)
      CALL COMP (0,1,1)
      CALL COMP (1,0,1)
      CALL COMP (1,1,1)
      DO 100 N=2,NMAX
      I=0
      J=0
      K=0
      CALL COMP (N,J,K)
      CALL COMP (I,N,K)
      CALL COMP (I,J,N)
      JMAX=N-1
      DO 200 J=1,JMAX
  200 CALL COMP (N,J,K)
      DO 300 I=1,N
  300 CALL COMP (I,N,K)
      KMAX=N-1
      I=0
      DO 400 K=1,KMAX
  400 CALL COMP (I,N,K)
      JMAX=N
      DO 500 J=1,JMAX
  500 CALL COMP(I,J,N)
      J=0
      KMAX=N-1
      DO 600 K=1,KMAX
  600 CALL COMP(N,J,K)
      IMAX=N
      DO 700 I=1,IMAX
  700 CALL COMP(I,J,N)
      JMAX=N-1
      DO 900 K=1,KMAX
      DO 800 J=1,JMAX
  800 CALL COMP(N,J,K)
  900 CONTINUE
      KMAX=N-1
      DO 1200 K=1,KMAX
      DO 1000 I=1,N
      IF (K.EQ.KMAX) GO TO 1
 1000 CALL COMP(I,N,K)
      GO TO 1200
    1 IMAX=N-1
      DO 1100 I=1,IMAX
 1100 CALL COMP(I,N,K)
 1200 CONTINUE
      K=N
```

```
      JMAX=N-1
      DO 1500 J=1,JMAX
      DO 1300 I=1,N
      IF (J.EQ.JMAX) GO TO 2
 1300 CALL COMP(I,J,K)
      GO TO 1500
    2 IMAX=JMAX
      DO 1400 I=1,IMAX
      CALL COMP(I,J,K)
      JMAX=N
      IF (I.EQ.IMAX) GO TO 1600
      CALL COMP(I,JMAX,K)
 1400 CONTINUE
 1500 CONTINUE
 1600 CONTINUE
      M=N-1
      CALL COMP(N,N,M)
      CALL COMP(M,N,N)
      CALL COMP(N,M,N)
      CALL COMP(N,N,N)
  100 CONTINUE
 1700 CONTINUE
 1800 WRITE (6,350)
      STOP
      END

      SUBROUTINE COMP(M1,M2,N)
      DIMENSION W(11,11,11)
      COMMON A1,A2,B1,B2,P,Q,R,W
  350 FORMAT (10X,I2,2(3X,I2),7X,F10.2,11X,F4.2)
      IF ((M1.EQ.0).AND.(M2.GT.0).AND.(N.GT.0)) GO TO 1
      IF ((M1.GT.0).AND.(M2.EQ.0).AND.(N.GT.0)) GO TO 2
      IF ((M1.EQ.0).AND.(M2.EQ.0).AND.(N.GT.0)) GO TO 3
      IF (N.EQ.0) GO TO 4
      TEST1=(A1*W(M1,M2+1,N+1)-A2*W(M1+1,M2,N+1))*(B1*M1+B2*
     1M2+A2*N)
      TEST2=(A1-A2)*((B1*M1+B2*M2)*W(M1+1,M2+1,N)+A2*N*
     1W(M1+1,M2,N+1))
      PHI=0.0
      IF (TEST1.GT.TEST2) PHI=1.0
      DENOM=PHI*A1*N+(1.0-PHI)*A2*N +B1*M1+B2*M2
      W(M1+1,M2+1,N+1)=(PHI*A1*N*W(M1,M2+1,N+1)+(1.0-PHI)*A2
     1*N*W(M1+1,M2,N+1)+(B1*M1+B2*M2)*W(M1+1,M2+1,N))/DENOM
      WRITE (6,350) M1,M2,N,W(M1+1,M2+1,N+1),PHI
      RETURN
    1 W(1,M2+1,N+1)=(B2*M2*W(1,M2+1,N)+A2*N*W(1,M2,N+1))/
     1(B2*M2+A2*N)
      RETURN
    2 W(M1+1,1,N+1)=(B1*M1*W(M1+1,1,N)+A1*N*W(M1,1,N+1))/
     1(B1*M1+A1*N)
      RETURN
    3 W(1,1,N+1)=R*N
      RETURN
    4 W(M1+1,M2+1,1)=-(P*M1+Q*M2)
      RETURN
      END
```

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